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An algebraic model of Coulomb scattering with spin

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Abstract

A new matrix-valued realization for the $so(3, 1)$ algebra leads to a natural generalization of the Coulomb scattering problem of a particle with spin. The underlying $su(2)$ gauge structure of this realization recasts the scattering problem into a familiar form, namely, the Coulomb scattering problem of a *collection* of dyons (particles having both electric and magnetic charges). Using this equivalent form and the results of Zwanziger for such systems, the scattering matrix can be calculated in the helicity formalism.

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1. Introduction

The dynamical symmetry of the three-dimensional Coulomb problem is one of the best studied examples illustrating the power of group theoretical methods. Since the seminal work of Pauli [1], Fock [2] and Bargmann [3] we have learnt that the bound state problem can be described using the symmetry group $SO(4)$. The description of the less obvious scattering states was given by Zwanziger [4] using the non-compact group $SO(3, 1)$, i.e. the Lorentz-group. After the advent of algebraic scattering theory (AST) [5] the scattering problem on a Coulomb potential was also reformulated in a purely algebraic language. In AST one identifies the symmetry of the interaction free (asymptotic) region, and then using the theory of group contractions and expansions relates this asymptotic symmetry to the symmetry of the scattering problem with some interaction term. After group theoretical manipulations by introducing the so-called Euclidean connection [6, 7] the functional form of the scattering matrix can be obtained. In this approach *no* explicit coordinate realization for the generators and for the interaction term is needed. In the case of the Coulomb problem the symmetry group in question is $SO(3, 1)$ and the asymptotic symmetry group is $E(3)$, the three-dimensional Euclidean group. The functional form of the *general* $SO(3, 1)$ scattering matrix calculated in this way gives rise to the one of Coulomb scattering after a proper identification of an irreducible representation describing the scattering process. This identification relates the label(s) of the irrep to the scattering energy. Other irreps describe scattering situations different from the Coulomb one but still having $SO(3, 1)$ symmetry.

However, there are some problems with this scheme. One of the most important problems to be solved is the question of how to incorporate spin degrees of freedom into this formalism. We expect that the addition of spin will lead to additional restrictions on the parameters appearing in the functional form of the scattering matrix. AST somehow has to be appropriately modified to tackle such situations.

The aim of this paper is twofold. Firstly we would like to present a generalization of the Coulomb scattering problem with spin degrees of freedom also present. The symmetry group in question is still $SO(3, 1)$. Secondly, we would like to see the possible restrictions that the inclusion of spin effects has on the parameters in the scattering matrix. This goal is achieved by presenting an explicit matrix-valued realization for the $so(3, 1)$ algebra. Though the use of explicit realizations is contrary to the spirit of AST, via the use of this realization we can calculate the S-matrix and demonstrate the nature of restrictions AST ought to cope with. In this way, besides providing an interesting generalization of the Coulomb problem, this new exactly solvable model, by giving us new hints, may pave the way for a generalized AST capable of incorporating spin degrees of freedom as well.

The organization of this paper is as follows. In section 2 we give a brief review of the usual Coulomb scattering problem with symmetry group $SO(3, 1)$. The next section is devoted to the construction of our matrix-valued realization for the 'spinning Coulomb problem', i.e. the scattering problem of a charged particle with spin s on a Coulomb potential. In section 4 by using the underlying gauge structure of our realization, we diagonalize our matrix-valued generators. In this way our scattering channels are completely decoupled. Moreover, this gauge transformed form reveals a structure which is well known to physicists. It describes the multichannel scattering problem of $(2s+1)$ dyons (i.e. particles with both electric and magnetic charges) on a Coulomb potential. Since the related problem for one channel has already been solved by Zwanziger [12] we merely have to use his results. This is done in section 5. Here we also present some hints for an alternative derivation. The conclusions and some comments are left for section 6.

2. The Coulomb problem

The basic nonrelativistic Coulomb scattering problem is epitomized by the Hamiltonian

$$H = \frac{P^2}{2} - \frac{\alpha}{R} \tag{1}$$

in which α is the Coulomb coupling constant. Throughout this paper we use the system of units in which $m = \hbar = 1$. This Hamiltonian supports an $so(3, 1)$ group algebra with the Casimir operators being the angular momentum operator $\{\boldsymbol{L}\}$ and the Runge–Lenz vector $\{\boldsymbol{K}\}$. Those operators are defined by

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} \tag{2}$$

and

$$\boldsymbol{K} = \tfrac{1}{2}\,[\boldsymbol{P} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{P}] - \alpha \boldsymbol{n} \tag{3}$$

where $\boldsymbol{n} \equiv \boldsymbol{R}/R$, and they satisfy commutation relations

$$[L_i, L_i] = \mathrm{i}\epsilon_{ijk} L_k \qquad [L_i, K_j] = \mathrm{i}\epsilon_{ijk} K_k \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk} L_k (2H). \tag{4}$$

Furthermore, for the Coulomb problem, $\boldsymbol{L}$ and $\boldsymbol{K}$ are conserved quantities since

$$[H, \boldsymbol{L}] = [H, \boldsymbol{K}] = 0. \tag{5}$$

Notice however, that the commutation relations given in equation (4) are *not* the defining relations of a Lie algebra. Rather they identify an algebra in which the structure constants also depend on the generators, but by restricting all the operators to subspaces of *fixed* energy, a Lie-algebra results. Thus as we consider the scattering problem at fixed (positive) energy $E = k^2/2$, by introducing new generators $\boldsymbol{K}^{(k)} \equiv \boldsymbol{K}/k$ for each value of the scattering energy, the $(\boldsymbol{L}, \boldsymbol{K}^{(k)})$ pair satisfy an $so(3, 1)$ algebra. And as $so(3, 1)$ is a Lie algebra of rank two, we have two independent Casimir operators, C_1 and C_2, commuting with those generators. Specifically,

$$\begin{aligned} C_1 &= \boldsymbol{L}^2 - \boldsymbol{K}^{(k)2} = -\left(\frac{\alpha}{k}\right)^2 - 1 \\ C_2 &= \boldsymbol{L} \cdot \boldsymbol{K}^{(k)} = \boldsymbol{K}^{(k)} \cdot \boldsymbol{L} = 0. \end{aligned} \tag{6}$$

3. The Coulomb problem with spin

To construct a generalization of the operators $\boldsymbol{L}$ and $\boldsymbol{K}$ incorporating spin, we seek a realization of the algebra given in equation (4) in terms of matrix-valued differential operators of $(2s + 1) \times (2s + 1)$ dimensions, where s is the spin. In that search it is natural to consider replacing $\boldsymbol{L}$ by $\boldsymbol{J}(= \boldsymbol{L} + \boldsymbol{S})$. The first question that arises, then, is: 'What is the matrix-valued form of the Runge–Lenz vector?'. Denoting this quantity by $\boldsymbol{M}$, the commutation relations that must then be satisfied are

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k \qquad [J_i, M_j] = \mathrm{i}\epsilon_{ijk} M_k \qquad [M_i, M_j] = -\mathrm{i}\epsilon_{ijk} J_k (2\mathcal{H}) \tag{7}$$

$\mathcal{H}$ being a yet unknown Hamiltonian of the 'spinning Coulomb problem'. Notice that the second of these commutation relations states that $\boldsymbol{M}$ acts as a *vector operator* under $\boldsymbol{J}$.

As a first candidate for $\boldsymbol{M}$ an obvious guess is to replace $\boldsymbol{L}$ by $\boldsymbol{J}$ in the expression, equation (3), for $\boldsymbol{K}$. However with this choice, the third commutator in equation (7) cannot be satisfied. There are terms in the right-hand side of that commutator relation, then, that are not proportional to L_k.

Another possibility is to modify the vector $\boldsymbol{P}$ as it appears in the expression for $\boldsymbol{K}$, by rendering it to a *covariant derivative*. In previous papers [8, 9], such was used to construct matrix-valued realizations for the algebras $so(n, 1)$, $so(2, 2)$, and $so(3, 2)$, and wherein it was shown that the theory of matrix-valued realizations for those groups could be cast in the language of the theory of induced representations. In our case the problem of finding matrix-valued realizations describing spin is equivalent to finding the generators of the induced representation for $so(3, 1)$ induced by the $(2s + 1)$-dimensional representation of the $su(2) \sim so(3)$ sub-algebra. Moreover, in [8] it was shown that the required generators can be written in a form containing covariant derivatives $\boldsymbol{D} = \boldsymbol{P} + \boldsymbol{A}$, where $\boldsymbol{A}$ is an $su(2) \sim so(3)$ Lie-algebra-valued non-Abelian gauge-field. Consequently, a reasonable ansatz for the form of our operators is

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{L} + \boldsymbol{S} \\ \boldsymbol{M} &= \tfrac{1}{2}(\boldsymbol{D} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{D}) - \alpha \boldsymbol{n} \end{aligned} \tag{8}$$

where $\boldsymbol{D} = \boldsymbol{P} + \boldsymbol{A}$. Here $\boldsymbol{A} = \sum_{i=1}^{3} \boldsymbol{A}^i S_i$ is an $su(2) \sim so(3)$ valued gauge-field, with the S_i being generators in the usual irreducible representation with spin s satisfying $[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k$. Furthermore we need $\boldsymbol{M}$ to behave as a vector operator as required by the second commutator of equation (7), so that $\boldsymbol{D}$ also behaves as a vector. Then we require $[J_i, D_j] = \mathrm{i}\epsilon_{ijk} D_k$. Hence $\boldsymbol{A}$ also has to be a vector operator under $\boldsymbol{J}$. The simplest ansatz for $\boldsymbol{A}$ satisfying these conditions is $\boldsymbol{A} = f(R)\boldsymbol{S} \times \boldsymbol{R}$, where $R = |\boldsymbol{R}|$ and f is an unknown function to be determined. With this choice the first two sets of the commutators of equation (7) are satisfied. To determine the unknown function $f(R)$ we can calculate $[M_i, M_j]$ and seek values of $f(R)$

and $\mathcal{H}$ with which a term of the form $-\mathrm{i}\epsilon_{ijk}J_k(2\mathcal{H})$ results. But f can be determined in an easier fashion.

The key feature in doing so is that, according to general theory [8], the operator $\boldsymbol{J}$ can be written in a gauge covariant form using the *same* $\boldsymbol{D}$. Thereby $\boldsymbol{J} = \boldsymbol{R} \times \boldsymbol{P} + \boldsymbol{S}$ equates to $\boldsymbol{J} = \boldsymbol{R} \times \boldsymbol{D} + \boldsymbol{\Phi}$ for some vector $\boldsymbol{\Phi}$. Since the expression for $\boldsymbol{J}$ is independent of R, its gauge covariant expression should be as well. Using the ansatz for $\boldsymbol{A}$ and comparing terms, it can be shown that

$$\boldsymbol{J} = \boldsymbol{R} \times \boldsymbol{D} + (\boldsymbol{S} \cdot \boldsymbol{n})\boldsymbol{n} \tag{9}$$

where

$$\boldsymbol{A} = \frac{1}{R^2}\boldsymbol{S} \times \boldsymbol{R}. \tag{10}$$

Using this form, the matrix-valued generators can be expressed as

$$\boldsymbol{J} = \tfrac{1}{2}(\boldsymbol{R} \times \boldsymbol{D} - \boldsymbol{D} \times \boldsymbol{R}) + (\boldsymbol{S} \cdot \boldsymbol{n})\boldsymbol{n} \tag{11}$$

and

$$\boldsymbol{M} = \tfrac{1}{2}(\boldsymbol{D} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{D}) - \alpha\boldsymbol{n} \tag{12}$$

with it being understood that

$$\boldsymbol{D} = \boldsymbol{P} + \frac{1}{R^2}\boldsymbol{S} \times \boldsymbol{R}. \tag{13}$$

Though these generator forms are similar in structure to those in equation (8), we prefer the former for reasons of simplicity of application.

A straightforward, though laborious, calculation shows that the commutation relations of equations (7) are satisfied when the modified Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\boldsymbol{D}^2 - \frac{\alpha}{R} + \frac{1}{2R^2}(\boldsymbol{S} \cdot \boldsymbol{n})^2. \tag{14}$$

One can also show that $[\mathcal{H}, \boldsymbol{J}] = [\mathcal{H}, \boldsymbol{M}] = 0$, hence these operators then define conserved quantities for the dynamics defined by $\mathcal{H}$. Moreover, using the explicit form for $\boldsymbol{A}$, an alternative form of the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}{P_R}^2 + \frac{\boldsymbol{J}^2}{2R^2} - \frac{\alpha}{R} \tag{15}$$

where $P_R = -\mathrm{i}(\partial_R + 1/R)$. Hence the only modification of the usual Coulomb problem is the replacement of L^2 by J^2 in the centrifugal term.

As in the case of the usual Coulomb problem, we restrict the domain of definition of the generators to a subspace of fixed scattering energy $E = k^2/2$ of the Hamiltonian $\mathcal{H}$. Introducing the renormalized generators $\boldsymbol{M}^{(k)} \equiv \boldsymbol{M}/k$ the commutators for $\boldsymbol{J}$ and $\boldsymbol{M}^{(k)}$ are those of an $so(3, 1)$ algebra, for which there are two Casimir operators

$$\mathcal{C}_1 = \boldsymbol{J}^2 - {\boldsymbol{M}^{(k)}}^2 = (\boldsymbol{S} \cdot \boldsymbol{n})^2 - \left(\frac{\alpha}{k}\right)^2 - 1 \tag{16}$$

and

$$\mathcal{C}_2 = \boldsymbol{J} \cdot \boldsymbol{M}^{(k)} = -\frac{\alpha}{k}(\boldsymbol{S} \cdot \boldsymbol{n}). \tag{17}$$

Notice that $\mathcal{C}_2$ is proportional to the component of the spin along the direction of motion of the particle and so we interpret the eigenvalue λ of the quantity $-\boldsymbol{S} \cdot \boldsymbol{n}$ as an *helicity*. That helicity can take the values $-s \leqslant \lambda \leqslant s$ where s is the spin of the particle. Also as $[\boldsymbol{J}, \boldsymbol{S} \cdot \boldsymbol{n}] = 0$, the Hamiltonian commutes with $\boldsymbol{S} \cdot \boldsymbol{n}$. Hence helicity is a good quantum number characterizing the scattering process.

According to the representation theory of $SO(3,1)$, the irreducible unitary representations of $SO(3,1)$ capable of describing scattering states are labelled by the pair of numbers (j_0, j_1) where $j_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $j_1 = \mathrm{i}c$ where $c \in \mathbb{R}$. These are the unitary representations in the principal series [10]. Then the Casimir operators acting on the scattering states, labelled as $|j_0, j_1\rangle$, satisfy eigenvalue equations,

$$\mathcal{C}_1|j_0, j_1\rangle = ({j_0}^2 + {j_1}^2 - 1)|j_0, j_1\rangle \tag{18}$$

and

$$\mathcal{C}_2|j_0, j_1\rangle = -\mathrm{i}j_0 j_1|j_0, j_1\rangle. \tag{19}$$

Consistent with the definitions, equations (16) and (17), the scattering states are

$$|j_0, j_1\rangle = \left||\lambda|, \frac{\alpha}{k}\mathrm{sgn}\lambda\right\rangle \qquad -s \leqslant \lambda \leqslant s \tag{20}$$

i.e. are helicity eigenstates. They span the irreducible unitary representation space of the $SO(3,1)$ group.

We alternatively can describe the space of scattering states by extending the group $SO(3,1)$ by including also the space reflections. For this group the representation space is spanned by the direct sum of the spaces:

$$\left|-\lambda, \frac{\alpha}{k}\right\rangle \oplus \left|\lambda, \frac{\alpha}{k}\right\rangle. \tag{21}$$

These subspaces are mapped to each other by the operator of space reflections. According to the general theory of invariants for this extended group there also exists, besides the second order $\mathcal{C}_1$, a *fourth order* Casimir operator, C_4. However, these operators cannot discriminate between the mirror conjugated states of equation (21). In order to establish a one-to-one correspondence between the labelling of states and the spectra of invariant operators it is sufficient to replace C_4 by $\mathcal{C}_2$. This justifies our use of $\mathcal{C}_2$ since this operator, being a pseudoscalar one, discriminates between the mirror conjugated states. Since the states $|-\lambda, \frac{\alpha}{k}\rangle$ and $|\lambda, -\frac{\alpha}{k}\rangle$ are equivalent [10] we can use both equations (20) and (21) as representations of our scattering states.

We have managed to describe scattering states group theoretically but, for this description to make sense, a physical meaning of the diagonalization of the operator $-\boldsymbol{S}\cdot\boldsymbol{n}$ must be specified. This will be discussed in the next section.

4. Gauge transformations

A basic entity underlying this construction is the $su(2)$-valued gauge-field

$$\boldsymbol{A} = \frac{1}{R^2}\boldsymbol{S}\times\boldsymbol{R}. \tag{22}$$

Its presence in generators ensures that they transform covariantly under $su(2)$ gauge transformations. To study these gauge transformations it is useful to introduce the one-form $A = \boldsymbol{A}(\boldsymbol{R})\cdot \mathrm{d}\boldsymbol{R}$ which, since $\mathrm{d}\boldsymbol{R} = \boldsymbol{n}\,\mathrm{d}R + R\,\mathrm{d}\boldsymbol{n}$, can be written in the form

$$A = \boldsymbol{S}\cdot(\boldsymbol{n}\times \mathrm{d}\boldsymbol{n}). \tag{23}$$

This one-form does not depend on R. It merely depends on the unit vector $\boldsymbol{n}$ which can be parametrized by the local coordinates (θ, φ) of the sphere S^2. A is thus a one-form residing on S^2 so that the local gauge transformations of interest can be parametrized by the points of S^2. Hence the general form of these transformations is

$$A' = U^\dagger A U - \mathrm{i}U^\dagger \mathrm{d}U \qquad \text{where} \quad U \in SU(2) \tag{24}$$

and, with $|\boldsymbol{m}| = 1$,

$$U(\theta, \varphi) = \exp\left[\mathrm{i}\beta(\theta, \phi)\boldsymbol{m}(\theta, \phi) \cdot \boldsymbol{S}\right]. \tag{25}$$

Notice that the transformation is solely a *local* one over S^2.

It is instructive to calculate the two-form (field strength of $\boldsymbol{A}$) $F = \mathrm{d}A + \mathrm{i}A \wedge A$. The result is

$$F = \tfrac{1}{2}\epsilon_{ijk} B_i \, \mathrm{d}R_j \wedge \mathrm{d}R_k \tag{26}$$

where

$$\boldsymbol{B} = \frac{\boldsymbol{n}}{R^2}(\boldsymbol{S} \cdot \boldsymbol{n}) \tag{27}$$

acts like a non-Abelian magnetic monopole with strength associated with the operator $\boldsymbol{S} \cdot \boldsymbol{n}$. Then, as $\boldsymbol{B}$ transforms covariantly under $SU(2)$ gauge transformations as

$$\boldsymbol{B}' = U^\dagger \boldsymbol{B} U \tag{28}$$

the requirement for the diagonalization of $\boldsymbol{S} \cdot \boldsymbol{n}$ also is a gauge transformation. This *local* diagonalization scheme will turn $\boldsymbol{B}$ to a collection of $(2s + 1)$ ordinary (Abelian) magnetic monopoles with pole strengths given by the helicities. The fact that there is no *global* transformation of this kind over S^2 reflects the truly non-Abelian nature of the gauge-field of equation (26). Note also that the last term in the right-hand side of equation (11) is $R^2\boldsymbol{B}$ which transforms covariantly. Thus the terms containing $\boldsymbol{D}$ do so as well and thus the generators $\boldsymbol{J}$ and $\boldsymbol{M}$ are gauge covariant.

To determine the gauge transformation, we make use of

$$U^\dagger \boldsymbol{S} U = \cos\beta \boldsymbol{S} + (1 - \cos\beta)\boldsymbol{m}(\boldsymbol{m} \cdot \boldsymbol{S}) + \sin\beta \boldsymbol{S} \times \boldsymbol{m} \tag{29}$$

and

$$-\mathrm{i}U^\dagger \mathrm{d}U = (\boldsymbol{m} \cdot \boldsymbol{S})\, \mathrm{d}\beta + \sin\beta(\mathrm{d}\boldsymbol{m} \cdot \boldsymbol{S}) + (1 - \cos\beta)(\boldsymbol{m} \times \mathrm{d}\boldsymbol{m}) \cdot \boldsymbol{S} \tag{30}$$

which are special cases of the formulae developed previously [8] in a more general context.

It is easy to check that the choice

$$\beta = \theta \qquad (m_1, m_2, m_3) = (\sin\varphi, -\cos\varphi, 0) \tag{31}$$

defining $U(\theta, \phi)$ per equation (25), not only diagonalizes $\boldsymbol{S} \cdot \boldsymbol{n}$ but also effects a transform A' that gives an Abelian gauge-potential, i.e.

$$U^\dagger \boldsymbol{S} \cdot \boldsymbol{n} U = S_3 \tag{32}$$

and

$$A' = U^\dagger A U - \mathrm{i}U^\dagger\, \mathrm{d}U = (1 - \cos\theta)\, \mathrm{d}\varphi\, S_3. \tag{33}$$

In this gauge the generators $\boldsymbol{J}$ and $\boldsymbol{M}$ are *diagonal* matrices of the form

$$\boldsymbol{J}' \equiv U^\dagger \boldsymbol{J} U = \tfrac{1}{2}(\boldsymbol{R} \times \boldsymbol{D}' - \boldsymbol{D}' \times \boldsymbol{R}) + S_3 \boldsymbol{n} \tag{34}$$

and

$$\boldsymbol{M}' \equiv U^\dagger \boldsymbol{M} U = \tfrac{1}{2}(\boldsymbol{D}' \times \boldsymbol{J}' - \boldsymbol{J}' \times \boldsymbol{D}') - \alpha \boldsymbol{n} \tag{35}$$

where

$$\boldsymbol{D}' = \boldsymbol{P} + \boldsymbol{A}' \tag{36}$$

with $\boldsymbol{A}'$ calculated from equation (33):

$$\boldsymbol{A}' = \frac{1}{R(R + R_3)} \begin{pmatrix} -R_2 \\ R_1 \\ 0 \end{pmatrix} S_3. \tag{37}$$

Of course the process of diagonalization of $\boldsymbol{S}\cdot\boldsymbol{n}$ via a gauge transformation is not unique. Indeed using

$$U \mapsto U\mathrm{e}^{\mathrm{i}\gamma S_3} \qquad \gamma \equiv \gamma(\theta,\varphi) \tag{38}$$

the resulting transformation still effects diagonalization. But it leads to a gauge-field

$$\boldsymbol{A}'' = \boldsymbol{A}' + \mathrm{d}\gamma\, S_3. \tag{39}$$

For example with the choice $\gamma = -2\varphi$ one finds

$$\boldsymbol{A}'' = -(1+\cos\theta)\,\mathrm{d}\varphi\, S_3. \tag{40}$$

By reverting to Cartesian coordinates, equation (33) equates to a collection of ordinary magnetic monopole gauge-potentials with monopole strengths given by the helicities $-s \leqslant \lambda \leqslant s$ and which are singular on the negative z-axis (see also equation (37)). Hence these fields are analytic merely on the patches homeomorphic to the northern hemispheres of the sphere S^2. Similarly, equation (40) equates to such a collection which is singular on the positive z-axis with fields analytic in the southern hemisphere of the sphere S^2. By comparison the gauge-field of equation (23) is analytic over all of S^2.

Thus our algebraic model can relate to a physical problem and one well known to physicists, namely, the scattering of dyons (particles with both electric and magnetic charges) in a Coulomb field. However, in contrast to the usual treatment that involved only one dyon, with this approach one can entertain a *collection* of dyons. Moreover, the *Dirac-quantized values* for the monopole charges are intrinsically related to the components of the *spin* along the direction of motion of the particle, i.e. the helicities. We have also seen that the gauge transformed realization (where the channels are completely decoupled) whilst being a more transparent approach from the physical point of view, cannot make sense globally. On the other hand our original realization, epitomized by equations (8)–(13), is *global* but the channels are coupled.

5. The scattering matrix

In the preceding section we demonstrated that, after diagonalization of the matrix-valued operators $\boldsymbol{J}$ and $\boldsymbol{M}$, it was feasible to transform the model of Coulomb scattering with spin to a gauge equivalent form describing the Coulomb potential scattering of $(2s+1)$ *uncoupled* dyons of monopole charge λ and spin s; the $(2s+1)$ scattering channels being labelled by the helicity eigenvalues $-s \leqslant \lambda \leqslant s$. We note that the *one* channel problem of scattering of individual dyons on a Coulomb potential has been investigated by Zwanziger [12] and so, not only do we have his results for confirmation, we can also suggest an alternative algebraic derivation of them based on the theory of induced representations.

First notice that for the global realization per equation (8), $\boldsymbol{J}\cdot\boldsymbol{n} = \boldsymbol{S}\cdot\boldsymbol{n}$. This quantity is proportional to C_2 and commutes with the Hamiltonian of equation (15). Thus its eigenvectors characterize incident and emergent particle scattering states, $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$, respectively. These states tend to wavepackets as $t \to -\infty$ and $t \to \infty$ directed along the $-\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}}$ directions, respectively. Here $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. Thus these states tend to the eigenstates of the operator $\boldsymbol{n}$ with eigenvalues -1 for the 'in' and $+1$ for the 'out' cases, respectively. Hence

$$(-\boldsymbol{S}\cdot\boldsymbol{n})\,|\boldsymbol{k};\mathrm{in/out}\rangle = \pm\lambda\,|\boldsymbol{k};\mathrm{in/out}\rangle\,. \tag{41}$$

Using the representation theory of the $so(3,1)$ algebra, Zwanziger [12] has shown that the scattering matrix

$$S(\boldsymbol{k}',\boldsymbol{k}) \equiv \langle\boldsymbol{k}';\mathrm{out}|\boldsymbol{k};\mathrm{in}\rangle \tag{42}$$

has the form

$$S(\boldsymbol{k}', \boldsymbol{k}) = \sum_{j \geqslant |\lambda|} (2j+1) e^{2i\delta_j} \mathcal{D}^j_{-\lambda,\lambda}(g^{-1}_{\boldsymbol{k}'} g_{\boldsymbol{k}}) \tag{43}$$

where

$$\hat{\boldsymbol{k}} = g_{\boldsymbol{k}} \hat{\boldsymbol{z}} \qquad g_{\boldsymbol{k}} \in SO(3) \tag{44}$$

and:

$$e^{2i\delta_j} = \frac{\Gamma(j+1-i\alpha/k)}{\Gamma(j+1+i\alpha/k)} \qquad j = |\lambda|, |\lambda|+1, \ldots. \tag{45}$$

Thus δ_j have the same form as the Coulomb phase shifts. But the Hamiltonian, equation (15), differs from that of the usual Coulomb problem by the replacement of $\boldsymbol{L}$ with $\boldsymbol{J}$ in the centrifugal term. In addition, there is now a *restricted set* of allowed values for j and, unlike the results of Zwanziger's study, there is now a *collection* of such systems where $-s \leqslant \lambda \leqslant s$. There are $(2s+1)$ phase shifts of the form equation (45), and the corresponding scattering matrix has a $(2s+1) \times (2s+1)$ diagonal form. The scattering channels are labelled by the different possible values of the helicity λ; the exact values of which are set by the restrictions on the allowed values for j in equation (45). For $\lambda = 0$, when the particle has integer spin, ordinary Coulomb scattering results but with j used instead of l. Note that the restriction on j depends solely on $|\lambda|$.

Following Zwanziger [12], we can calculate the scattering amplitude. Some subtle issues about phase conventions then arise as is usual when one considers scattering processes described in the helicity formalism. However, as the properties of the scattering matrix in this formalism were discussed in detail in [12], we do not consider those points further, especially as they play no role in calculation of the differential cross section. To specify that cross section, it is convenient to revert to a Cartesian representation where $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{k}}' = \cos\theta\,\hat{\boldsymbol{z}} + \sin\theta\,\hat{\boldsymbol{x}}$. In this frame

$$\mathcal{D}^j_{-\lambda,\lambda}(g^{-1}_{\boldsymbol{k}'} g_{\boldsymbol{k}}) = d^j_{-\lambda,\lambda}(\theta) \tag{46}$$

and with $f(\boldsymbol{k}', \boldsymbol{k}) = S(\boldsymbol{k}', \boldsymbol{k})/2ik$ for $\theta \neq 0$ being the scattering amplitude, helicity cross sections are [12]

$$\sigma(k, \theta, \lambda) = |f(\boldsymbol{k}', \boldsymbol{k})|^2 = \frac{\lambda^2 + (\alpha/k)^2}{k^2(1-\cos\theta)^2} \tag{47}$$

which, on averaging over the $(2s+1)$ possible helicity states, gives the differential cross section as

$$\left\langle \frac{d\sigma}{d\Omega}(E, \theta, s) \right\rangle = \frac{\alpha^2}{16E^2 \sin^4\frac{\theta}{2}} \left(1 + \frac{2}{3} s(s+1) \frac{E}{\alpha^2} \right). \tag{48}$$

Hence the differential cross section retains the Rutherford form but with an energy dependence modified by a linear term whose slope depends on the spin.

An alternative algebraic derivation can be made utilizing the Frobenius reciprocity theorem of the theory of induced representations. Consider the chain of algebras $so(2) \subset so(3) \subset so(3, 1)$. A matrix-valued generator of $so(2)$ is $J_3 = L_3 + S_3$, and so we induce a representation for $so(3)$ starting from the matrix-representation of $so(2)$. Since this algebra is Abelian, it has one-dimensional irreducible representations. Using S_3 as the generator of the inducing representation means that we use a *reducible* representation with individual one-dimensional irreducible representations labelled by $-s \leqslant \lambda \leqslant s$. As has been shown [8], the generators of the induced representation for $so(3)$ are precisely the $\boldsymbol{J}'$ of equation (34) with the gauge-potential of equation (33). Thus, after gauge transformation, it is precisely the quantity $\boldsymbol{J}'^2$ of the Hamiltonian, equation (15), and the eigenvalues are $j(j+1)$. However, the range of allowed values for j is restricted. According to Frobenius reciprocity (see [8] and references therein), the allowed values for j are those set by the restriction that the corresponding $so(3)$ representation restricted to $so(2)$ contains the Abelian representation labelled by λ. Restricting a $(2j+1) \times (2j+1)$-dimensional matrix representation to $so(2)$ gives the matrix with diagonal entries, $-j, -j+1, \ldots, j$. Clearly among them the value λ occurs only when $j = |\lambda|, |\lambda| + 1 \ldots$; precisely the restriction implicit in equation (45).

With this (induced) representation for the $so(3)$ generators of equation (34), those generators can be used as the starting point in a further inducing process, by which means one can induce a representation for $so(3, 1)$. That was how generators $\boldsymbol{M}'$ were built with the embedding of generators $\boldsymbol{J}'$. Of course in this case the representation for $so(3, 1)$ labelled by (j_0, j_1) when restricted to the $so(3)$ sub-algebra again contains precisely the j values $j_0, j_0 + 1, \ldots$ where $j_0 \equiv |\lambda|$ according to equation (20). Notice that a similar construction can be found [9] for the chain of algebras $so(3) \subset so(3, 1) \subset so(3, 2)$.

6. Conclusions

In this paper an algebraic model with a dynamical symmetry group $SO(3, 1)$ for the Coulomb scattering problem of a charged particle with spin s was given. For the construction of this model we introduced a new realization for the $so(3, 1)$ algebra in terms of $(2s+1) \times (2s+1)$-dimensional matrix-valued differential operators. The six generators of $so(3, 1)$ are the three components of the total angular momentum $\boldsymbol{J}$, and the three components of a suitable matrix-valued generalization of the Runge–Lenz vector $\boldsymbol{M}$. The Hamiltonian of the system is just the usual Coulomb Hamiltonian with $\boldsymbol{L}^2$ replaced by $\boldsymbol{J}^2$ in the centrifugal term. Exploiting the underlying $SU(2)$ gauge structure of our realization we have shown how to transform this realization to a diagonal one in spin space. In the gauge transformed realization the channels are decoupled, and the resulting $(2s + 1)$ one-channel problems are well known to physicists. By denoting by λ $(-s \leqslant \lambda \leqslant s)$ the possible values for the helicities of our particle, these are scattering problems of dyons with monopole charge λ on a Coulomb field. Since this problem has already been solved by Zwanziger [12], we merely had to use his result for the calculation of the scattering quantities. Motivated by deep results of group theory we have also given hints for an alternative derivation of Zwanziger's results.

According to the results of our paper the Coulomb phase shifts retain their functional form as fixed by AST [5] with l replaced by j, where of course $|l - s| \leqslant j \leqslant l + s$. The important result of this paper is that *the allowed values for* j *are further restricted* by the condition $j = |\lambda|, |\lambda| + 1, \ldots$, with the values of λ labelling the different helicity channels. Hence we may conclude that in order to account for such possible restrictions arising in an algebraic description of scattering problems with spin AST has to be appropriately modified.

We remark in closing that the *global* gauge-field of equation (10) which is at the heart of our construction is well known to gauge-field theorists, and is also a solution to the $SU(2)$ Yang–Mills equations [11]. On the other hand the *local* gauge-field of equation (37), which is a collection of monopole ($U(1)$) gauge-fields obtained after the gauge transformation, is precisely of the form usually induced in the effective nuclear Hamiltonian in the Born–Oppenheimer treatment of diatomic molecules [13]. Hence, in principle, our Hamiltonian given in equation (15) should also be able to emerge as an effective Hamiltonian from coupled systems of slow and fast variables, after averaging with respect to the fast degrees of freedom. Of course we can use both of our gauge-fields for such considerations. Note, however, that these gauge-fields living on S^2 can be defined globally as $SU(2)$ but merely locally as a collection of $U(1)$ gauge-fields. As can be shown, this statement is a consequence of the fact that unlike $U(1)$, $SU(2)$ is simply connected.

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